The discrete joint probability distribution \( P \) over data from two classes, and the neighborhood function \( N(\cdot) \) define a bipartite conflict graph \( G \) with incidence matrix \( E \). Let \( p \in \mathbb{R}^V \) with \( p_v = \mathbb{P}\{\cdot|v\} \). Let \( q^* \) be the minimizer of the following program:

\[
\min \quad q \\
\text{s.t.} \quad q \geq 0 \\
M q \leq 1.
\]

Then, there is a classifier \( h^* \) that achieves the correct-classification probabilities \( q^* \) and for all \( h \), \( \mathbb{E}_P[\mathbb{E}_v[CE(h, v)]] \leq \mathbb{E}_p[\mathbb{E}_v[CE(h, v)]] \).

**Takeaways:**
- Lower bound is achievable: the solution provides optimal classification probabilities.
- Solution to the dual convex problem is the optimal adversarial strategy.
- Applies to all discrete distributions, including popular vision datasets such as CIFAR-10 etc.

**Establishing lower bounds**

**Theorem**

Let \( q \in \mathbb{R}^V \) be the vector of correct-classification probabilities obtained by a classifier. The feasible set of such probabilities is

\[
\{q \in \mathbb{R}^V : q \geq 0, M q \leq 1\}
\]

where

\[
M = \begin{bmatrix} E \end{bmatrix} \in \mathbb{R}^{(x,y)} \times V \text{ and } E \in \mathbb{R}^{x \times V} \text{ is the edge incidence matrix of the conflict graph.}
\]

**Lemma**

\[
\text{Let } q \in \mathbb{R}^V \text{ be the vector of correct-classification probabilities obtained by a classifier. The feasible set of such probabilities is}
\]

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\{q \in \mathbb{R}^V : q \geq 0, M q \leq 1\}
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\]

**Takeaways:**
- This is the fractional vertex packing polytope of the graph.
- Larger budgets lead to more intersections and a smaller feasible set.
- Construction of the conflict graph and associated polytope can be applied to any possible adversarial constraint, including standard \( \ell_p \) ball constraints.